WHEN YOU NEED TO BE SURE

CyFlex® Knowledge Article

Critical Flow Venturi Calculations per 40CFR1065.640

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Contents

Overview

This task computes the mass and molar flow rates through a critical flow venturi using the method specified in 40CFR1065. It is based on the `critical_flow` task which has minor inconsistencies with the 1065 standard. Refer to *Computing Critical Air Mass Flow*.

Prior to configuring this task, read and understand sections 40CFR1065.640 and 40CFR1065.642. The standard allows the use of constant values for compressibility factor, molar mass, and the ratio of specific heats for dilution air and raw and diluted exhaust gas. Take particular care to use the same values and simplifying assumptions that were used when the CFV calibration was performed.

The process described below for starting various auxiliary tasks in the `go.scp` assumes none of the simplifying assumptions have been made and the molar mass and ratio of specific heats are to be the result of a real-time calculation. To be consistent with 1065, the 1065 dry air (`comp.1065_AIR`) composition should be selected instead of the Harold Weber specified 'Cummins Air' (`comp.COMBUSTION_AIR`) whenever ambient air enters into the calculation.

The `cfv_1065` task is normally started in the `go` script that starts CyFlex.

Prerequisites

Prior to running `cfv_1065`, several other programs must also be running or have run. They include:

- `init_properties`
- `init_composition`
- `gas_prop`
- `add_water`

Refer to the following sample startup sequence.

```
######################################################################
#  Example startup for critical flow venturi meters
#
######################################################################
#  "init_properties" and "init_composition" must precede the first
#  copy of "cfv_1065" so that the gas composition variable will be
#  initialized
#
# init_properties creates the memory for composition and property
# variables - required for "critical_flow" & "gas_prop"

init_properties

# init_compositon reads /specs/properties/comp_specs.NNN and
initializes
# the values of composition variables to the last value saved when
# running or those permanently defined by a comp.<STREAM> file  To be
# 1065 compliant, use comp.1065_AIR for dry air composition.

init_composition

# gas_prop computes the properties of the cfv gas stream

gas_prop 12 1000 /specs/properties/prop_specs.99 &

# add_water adds water vapor to the cfv gas stream  To be 1065
#compliant, use comp.1065_AIR for dry air composition input to
# add_water.

add_water 12 1000 /specs/properties/addwater_specs.99 &

# cfv_1065 is used to compute the molar and mass flow rate through a
# critical flow venturi

cfv_1065   12   1000 /specs/cfv_specs.99 &
```

Running the Command

Refer to cyflex.com `cfv_1065` usage help for command syntax.

Specification File

The following is an example specification file used for a critical flow venturi.

```
# INPUTS
#
# barometer label
#               required value – Inlet and exit pressure are assumed to be
#               gage measurements
#
# inlet static gage pressure and temperature labels
#               required values – The label of a Cyflex variable or a
#               computed expression for the gage static pressure and
#               the label of a Cyflex variable for the temperature measured
#               at the inlet.  A computed expression for pressure may be
#               required if absolute pressure is being measured.
#
# throat diameter and inlet diameter
#               required values - These may be values, variable labels,
#               or computed expressions.
#
# exit static gage pressure
#               required value – Label of a Cyflex variable or a computed
#               expression containing the gage static pressure measured at
#               the exit of the critical flow venturi.  Used to verify that
#               the exit to inlet pressure ratio is within the allowable range
#               for constant discharge coeff.  A computed expression for
#               pressure may be required if absolute pressure is being
#               measured.
#
# specific heat ratio
#               required value – Ratio of specific heat for gas flowing
#               through the venturi.  Can be a value or a variable label.
#               See allowable simplifying assumptions in 40CFR1065.640©(5)
#
# molar mass
#               required value – Molar mass of the gas flowing through the
#               venturi.  Can be a value or a variable label.
#               See allowable simplifying assumptions in 40CFR1065.640©(5)
#
# compressibility factor
#               required value – Compressibility factor for gas flowing
#               through the venturi.  Can be a value or a variable label.
#               See allowable simplifying assumptions in 40CFR1065.640©(5)
#
# coefficient of discharge and maximum pressure ratio
#               required values – The coefficient of discharge for the venturi
#               and the maximum exit to inlet absolute static pressure ratio
#               for which the coeff of discharge deviation is acceptable per
#               1065.640(e). Can be values or variable labels.
#
# NOTE – Using Cyflex variables for throat diameter, coeff of discharge and
#         maximum pressure ratio would allow one instance of cfv_1065 to
#         handle combinations of multiple venturis based on switching logic
#         in gen_labels tied to physical valving arrangements.
#
```

```
# OUTPUTS
#
# molar and mass flow rates
#              required values - Labels for Cyflex variables to contain the
#              calculated molar and mass flow rates.  These variables will be
#              created if they do not already exist.
#
# valid result flag
#              required value – Label for a Cyflex logical variable that
#              will indicate whether the maximum pressure ratio has been
#              exceeded or some other error has been encountered that would
#              make the flow rate calculations invalid.  The variable will
#              be created if it does not already exist.
#
# throat to inlet and exit to inlet absolute static pressure ratios
#              optional values – Labels for Cyflex variables that will
#              contain the calculated throat to inlet static pressure ratio
#              and the measured exit to inlet static pressure ratio.  The
#              variables will be created if they do not already exist.
#
# inlet Mach number
#              optional value – Label for a Cyflex variable that will
#              contain the inlet Mach number.  The variable will be created
#              if it does not already exist.
#
# absolute static to stagnation inlet pressure and temperature ratios
#              optional values – Labels for Cyflex variables that will
#              contain the absolute static to stagnation inlet pressure and
#              temperature ratios.  The variables will be created if they
#              do not already exist.
#
############################################################################

# INPUTS
#
# barometer label
  barometer

# inlet static gage pressure  inlet static temperature
  cfv_in_p                    cfv_in_t

# throat diameter  inlet diameter
  cfv_throat_dia   24.135[in]

# outlet static gage pressure
  cfv_ot_p

# specific heat ratio
  inlet_airP.GM

# molar mass
  inlet_airC.MM

# compressibility factor
  1.0[none]
```

```
# coefficient of discharge  maximum pressure ratio
  cfv_cd                    cfv_r_max
#
# OUTPUTS
#
# molar flow rate  mass flow rate
  cfv_nf           cfv_mf

# valid result flag
  cfv_valid

# absolute static pressure ratios
# throat to inlet          outlet/exit to inlet
  cfv_throat_in_p_ratio    cfv_exit_in_p_ratio

# inlet Mach number
  cfv_in_mach

# absolute static to stagnation ratios
# inlet pressure  inlet temperature
  cfv_in_stag_p_rat  cfv_in_stag_t_rat
```

Program Specifications

Critical Flow Venturi (CFV) calibration calculations are described in 40CFR1065.640. CFV molar flow calculations are described in 40CFR1065.642. The molar flow rate for a CFV is calculated using Equation 1065.642-4.

$$\dot{n} = C_d C_f \frac{A_t P_{in}}{\sqrt{Z M_{mix} R T_{in}}} \qquad \text{cfv.1}$$

where

$$\begin{aligned}
\dot{n} &= molar\ flow\ rate \\
C_d &= discharge\ coefficient \\
C_f &= flow\ coefficient \\
A_t &= venturi\ throat\ cross-sectional\ area \\
P_{in} &= inlet\ static\ absolute\ pressure \\
&= inlet\ static\ gage\ pressure + barometer \\
T_{in} &= inlet\ static\ absolute\ temperature \\
Z &= compressibility\ factor \\
M_{mix} &= molar\ mass\ of\ the\ gas\ mixture \\
R &= molar\ gas\ constant = 8.314472[J/(mol_K)]
\end{aligned}$$

The flow coefficient is calculated using Equation 1065.640-6

$$C_f = \left[\frac{2\gamma(r^{\frac{\gamma-1}{\gamma}} - 1)}{(\gamma-1)(\beta^4 - r^{\frac{-2}{\gamma}})} \right]^{\frac{1}{2}} \qquad \text{cfv.2}$$

where

$$\begin{aligned}
\gamma &= isotropic\ \mathrm{exp}onent\ (for\ ideal\ gases,\ ratio\ of\ specific\ heats\ for\ the\ gas\ mixture) \\
\beta &= ratio\ of\ venturi\ throat\ to\ inlet\ diameters \\
r &= ratio\ of\ throat\ to\ inlet\ static\ absolute\ pressures
\end{aligned}$$

The value for *r* is determined by solving Equation 1065.640-8, presented here in a slightly different form

$$f(r) = r^{\frac{1-\gamma}{\gamma}} + \left(\frac{\gamma-1}{2}\right)\beta^4 r^{\frac{2}{\gamma}} - \frac{\gamma+1}{2} = 0 \qquad \text{cfv.3}$$

We can use Newton's method to obtain an iterative solution for *r* by computing the derivative

$$f'(r) = \frac{1-\gamma}{\gamma} r^{\frac{1-2\gamma}{\gamma}} + \left(\frac{\gamma-1}{\gamma}\right)\beta^4 r^{\frac{2-\gamma}{\gamma}} \qquad \text{cfv.4}$$

and using

$$r_{i+1} = r_i - \frac{f(r_i)}{f'(r_i)} = r_i - \frac{r_i^{\frac{1-\gamma}{\gamma}} + \left(\frac{\gamma-1}{2}\right)\beta^4 r_i^{\frac{2}{\gamma}} - \frac{\gamma+1}{2}}{\frac{1-\gamma}{\gamma} r_i^{\frac{1-2\gamma}{\gamma}} + \left(\frac{\gamma-1}{\gamma}\right)\beta^4 r_i^{\frac{2-\gamma}{\gamma}}} \qquad \text{cfv.5}$$

Many of the terms remain constant and can be calculated outside the iterative loop. We can write

$$r_{i+1} = r_i - \frac{r_i^a + b\,r_i^c + d}{e\,r_i^f + g\,r_i^h} \qquad \text{cfv.6}$$

where

$$a = \frac{1-\gamma}{\gamma} \quad b = \left(\frac{\gamma-1}{2}\right)\beta^4 \quad c = \frac{2}{\gamma} \quad d = -\frac{\gamma+1}{2}$$
$$e = \frac{1-\gamma}{\gamma} \quad f = \frac{1-2\gamma}{\gamma} \quad g = \left(\frac{\gamma-1}{\gamma}\right)\beta^4 \quad h = \frac{2-\gamma}{\gamma} \qquad \text{cfv.7}$$

We iterate on the value of *r* until the change in the value from one iteration to the next becomes a very small percentage (0.001%) of the value. If the value does not converge within 20 iterations, set the status validation logical to false.

Once we have a value for *r*, we calculate the flow coefficient, C_f, and then the molar flow rate using the first two equations. Use an initial value of 0.5 for the first iteration and then use the previous value for subsequent passes.

If constant values are used for *γ* and *β*, both *r* and C_f will be constant and need only be calculated once.

We would also like to know the mass flow rate which is simply

$$\dot{m} = \dot{n}\,M_{mix} \qquad \text{cfv.8}$$

1065.640(e) specifies the process used for determining the value of the discharge coefficient and stipulates that the calculation is only valid up to the highest venturi pressure ratio, r_{CFV} , used in the calibration. The pressure ratio is given by Equation 1065.640-13, which in a slightly altered form is

$$r_{CFV} = 1 - \frac{\Delta P_{CFV}}{P_{in}} = \frac{P_{out}}{P_{in}} \qquad \text{cfv.9}$$

where

$$\Delta P_{CFV} = \textit{differential static pressure; inlet minus outlet}$$

$$P_{out} = \textit{venturi outlet static absolute pressure}$$

$$= \textit{venturi outlet static gage pressure} + \textit{barometer}$$

The instantaneous measured pressure ratio should be compared to the maximum value read from the input specification to determine the value to output to the status validation logical. As diagnostics, we also allow the option to output the values of *r* and r_{CFV} to Cyflex variables.

The Mach number at the inlet is another potentially useful diagnostic. Once we compute a value for *r*, the Mach number can be calculated using

$$M = \left[\left(r^{\frac{\gamma-1}{\gamma}}\frac{\gamma+1}{2} - 1\right)\frac{2}{\gamma-1}\right]^{\frac{1}{2}} \qquad \text{cfv.10}$$

Given the Mach number, we can then calculate the ratio of static to stagnation absolute pressure and temperature at the inlet to get a sense of how close the inlet conditions are to stagnation conditions. The ratio of static to stagnation absolute temperature at the inlet is given by

$$\frac{T_{in}}{T_o} = \left[1 + \frac{\gamma-1}{2}M^2\right]^{-1} \qquad \text{cfv.11}$$

The ratio of static to stagnation absolute temperature at the inlet is given by

$$\frac{P_{in}}{P_o} = \left[1 + \frac{\gamma-1}{2}M^2\right]^{\frac{-\gamma}{\gamma-1}} \qquad \text{cfv.12}$$

Derivation

1065 has us solving Equation 1065.640-8 for a pressure ratio that is not well defined. It is interesting to consider the source of that equation

Pressure Ratio

Starting with conservation of mass

$$\dot{m} = \rho V A = \rho^* V^* A^* \qquad \text{cfv.13}$$

where

$\dot{m} = mass\ \ flow\ rate$

$\rho = density$

$V = velocity$

$A = cross - sectional\ area$

The * superscript indicates critical flow conditions in the CFV throat. Variables that are not superscripted represent inlet conditions. Solving for the area ratio gives

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{V^*}{V} = \frac{\rho^*}{\rho}\frac{\sqrt{\gamma R_{mix} T^*}}{M\sqrt{\gamma R_{mix} T}} = \frac{1}{M}\frac{\rho^*}{\rho}\left(\frac{T^*}{T}\right)^{\frac{1}{2}} \qquad \text{cfv.14}$$

where

$R_{mix} = ideal\ gas\ constant\ of\ the\ mixture$

$M = inlet\ Mach\ number = \dfrac{V}{\sqrt{\gamma R_{mix} T}}$

For isentropic flow, the stagnation conditions are constant through the CFV, so

$$T_o = T^*\left[1+\frac{\gamma-1}{2}\right] = T\left[1+\frac{\gamma-1}{2}M^2\right]$$
$$\Rightarrow \frac{T^*}{T} = \left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right] \qquad \text{cfv.15}$$

and

$$\rho_o = \rho^*\left[1+\frac{\gamma-1}{2}\right]^{\frac{1}{\gamma-1}} = \rho\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{1}{\gamma-1}}$$

$$\Rightarrow \frac{\rho^*}{\rho} = \left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right]^{\frac{1}{\gamma-1}} \qquad \text{cfv.16}$$

Substituting into the area ratio equation gives

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right]^{\frac{1}{\gamma-1}}\left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right]^{\frac{1}{2}}$$

$$= \frac{1}{M}\left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad \text{cfv.17}$$

Squaring both sides of this equation and recognizing the relationship between the area and diameter ratios gives us

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{\beta^4} = \frac{1}{M^2}\left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right]^{\frac{\gamma+1}{(\gamma-1)}} \qquad \text{cfv.18}$$

But we are looking for a relationship between the diameter ratio and the pressure ratio, so for isentropic flow, we can write

$$P_o = P^*\left[1+\frac{\gamma-1}{2}\right]^{\frac{\gamma}{\gamma-1}} = P\left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}$$

$$\Rightarrow \left[\frac{P^*}{P}\right]^{\frac{\gamma-1}{\gamma}} = \left[\frac{1+\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}}\right] \qquad \text{cfv.19}$$

Substituting into the previous equation gives us

$$\frac{1}{\beta^4} = \frac{1}{M^2}\left[\frac{P^*}{P}\right]^{\frac{\gamma+1}{\gamma}} \qquad \text{cfv.20}$$

We still need to get rid of the Mach number, so we rearrange the pressure ratio equation to solve for the Mach number

$$M^2 = \left[\left[\frac{P^*}{P}\right]^{\frac{\gamma-1}{\gamma}}\frac{\gamma+1}{2} - 1\right]\frac{2}{\gamma-1} \qquad \text{cfv.21}$$

Which we can substitute into cfv.20 to get

$$\frac{1}{\beta^4} = \frac{\left[\frac{P^*}{P}\right]^{\frac{\gamma+1}{\gamma}}}{\left[\left[\frac{P^*}{P}\right]^{\frac{\gamma-1}{\gamma}}\frac{\gamma+1}{2} - 1\right]\frac{2}{\gamma-1}} \qquad \text{cfv.22}$$

Rearranging gets us to the desired result

$$\left[\left[\frac{P^*}{P}\right]^{\frac{\gamma-1}{\gamma}}\frac{\gamma+1}{2} - 1\right]\frac{2}{\gamma-1} = \beta^4\left[\frac{P^*}{P}\right]^{\frac{\gamma+1}{\gamma}}$$

$$\Rightarrow \left[\frac{P^*}{P}\right]^{\frac{1-\gamma}{\gamma}} + \frac{\gamma-1}{2}\beta^4\left[\frac{P^*}{P}\right]^{\frac{\gamma+1}{\gamma}}\left[\frac{P^*}{P}\right]^{\frac{1-\gamma}{\gamma}} - \frac{\gamma+1}{2}\left[\frac{P^*}{P}\right]^{\frac{\gamma-1}{\gamma}}\left[\frac{P^*}{P}\right]^{\frac{1-\gamma}{\gamma}} = 0 \qquad \text{cfv.23}$$

$$\Rightarrow \left[\frac{P^*}{P}\right]^{\frac{1-\gamma}{\gamma}} + \frac{\gamma-1}{2}\beta^4\left[\frac{P^*}{P}\right]^{\frac{2}{\gamma}} - \frac{\gamma+1}{2} = 0$$

So when we compare this result to cfv.3, we find that the r we are solving for turns out to be the CFV throat to inlet absolute static pressure ratio

$$r = \frac{P^*}{P} \qquad \text{cfv.24}$$

Mass Flow Rate

Now consider the equation for mass flow rate.

$$\dot{m} = \rho V A \quad \rho = \frac{P}{R_{mix} T} \quad V = Mc = M\sqrt{\gamma R_{mix} T}$$
$$\beta^2 = \frac{A^*}{A} \quad \Rightarrow \quad A = \frac{A^*}{\beta^2} \qquad \text{cfv.25}$$

Combining gives us

$$\dot{m} = \frac{P}{R_{mix} T} M \sqrt{\gamma R_{mix} T}\, \frac{A^*}{\beta^2} = \frac{PA^*}{\sqrt{R_{mix} T}} \frac{M\gamma^{\frac{1}{2}}}{\beta^2} \qquad \text{cfv.26}$$

Rearranging and taking the square root of equation cfv.21 gives us

$$M = \beta^2 \left[\frac{P^*}{P}\right]^{\frac{\gamma+1}{2\gamma}} \qquad \text{cfv.27}$$

Substituting into the mass flow rate equation gives

$$\dot{m} = \frac{PA^*}{\sqrt{R_{mix} T}} \left[\left[\frac{P^*}{P}\right]^{\frac{\gamma+1}{\gamma}} \gamma\right]^{\frac{1}{2}} = \frac{PA^*}{\sqrt{R_{mix} T}} \left[r^{\frac{\gamma+1}{\gamma}} \gamma\right]^{\frac{1}{2}} \qquad \text{cfv.28}$$

We recognize that the coefficient of discharge is to compensate for non-ideal behavior, but at first glance, the square root term is not obviously equivalent to the flow coefficient from 1065.640-6.

$$C_f = \left[\frac{2\gamma(r^{\frac{\gamma-1}{\gamma}} - 1)}{(\gamma - 1)(\beta^4 - r^{\frac{-2}{\gamma}})}\right]^{\frac{1}{2}} \qquad \text{cfv.29}$$

But this equation can be simplified significantly if we eliminate β using 1065.640-8. Solving 1065.640-8 for β gives us

$$\beta^4 = \frac{\frac{\gamma+1}{2} - r^{\frac{1-\gamma}{\gamma}}}{\left(\frac{\gamma-1}{2}\right) r^{\frac{2}{\gamma}}} \qquad \text{cfv.30}$$

The right-hand term in the denominator of 1065.640-6 simplifies to

$$\begin{aligned}
\beta^4 - r^{\frac{-2}{\gamma}} &= \frac{\frac{\gamma+1}{2} - r^{\frac{1-\gamma}{\gamma}}}{\left(\frac{\gamma-1}{2}\right) r^{\frac{2}{\gamma}}} - r^{\frac{-2}{\gamma}} \\
&= \frac{\frac{\gamma+1}{2} - r^{\frac{1-\gamma}{\gamma}} - r^{\frac{-2}{\gamma}}\left(\frac{\gamma-1}{2}\right) r^{\frac{2}{\gamma}}}{\left(\frac{\gamma-1}{2}\right) r^{\frac{2}{\gamma}}} \\
&= \frac{\frac{\gamma+1}{2} - r^{\frac{1-\gamma}{\gamma}} - \left(\frac{\gamma-1}{2}\right)}{\left(\frac{\gamma-1}{2}\right) r^{\frac{2}{\gamma}}} = \frac{1 - r^{\frac{1-\gamma}{\gamma}}}{\left(\frac{\gamma-1}{2}\right) r^{\frac{2}{\gamma}}}
\end{aligned} \qquad \text{cfv.31}$$

We can then simplify the entire denominator to get

$$(\gamma-1)(\beta^4 - r^{\frac{-2}{\gamma}}) = \frac{1 - r^{\frac{1-\gamma}{\gamma}}}{\left(\frac{\gamma-1}{2}\right) r^{\frac{2}{\gamma}}}(\gamma-1) = \frac{2\left(1 - r^{\frac{1-\gamma}{\gamma}}\right)}{r^{\frac{2}{\gamma}}} = \frac{2\left(1 - \frac{1}{r^{\frac{\gamma-1}{\gamma}}}\right)}{r^{\frac{2}{\gamma}}} = \frac{2\left(r^{\frac{\gamma-1}{\gamma}} - 1\right)}{r^{\frac{\gamma+1}{\gamma}}} \qquad \text{cfv.32}$$

Substituting that into the body of the equation gives us

$$\frac{2\gamma(r^{\frac{\gamma-1}{\gamma}} - 1)}{(\gamma-1)(\beta^4 - r^{\frac{-2}{\gamma}})} = \frac{2\gamma(r^{\frac{\gamma-1}{\gamma}} - 1) r^{\frac{\gamma+1}{\gamma}}}{2\left(r^{\frac{\gamma-1}{\gamma}} - 1\right)} = \gamma r^{\frac{\gamma+1}{\gamma}} \qquad \text{cfv.33}$$

So 1065.640-6 for a critical flow venturi can be simplified to

$$C_f = \left[\gamma r^{\frac{\gamma+1}{\gamma}} \right]^{\frac{1}{2}} \qquad \text{cfv.34}$$

Even though this is a simpler computation, we choose not to use it in the code for cfv_1065 since we would like it to be obviously consistent with 1065 by inspection of the code alone without reference to this document.

Compressibility

The `critical_flow` task, which was the predecessor to `cfv_1065` included an option to calculate the compressibility of air using the van der Waals equation. A brute force search was used to find an acceptable solution. It would have been more computationally efficient to use Newton's method to find an iterative solution. That process is documented here in case we decide to retain and improve the `critical_flow` task.

The van der Waals equation for compressibility is

$$f(Z_0) = Z_0^3 - \left(\frac{p_{0R}}{8T_{0R}} + 1 \right) Z_0^2 + \left(\frac{27}{64} \frac{p_{0R}}{T_{0R}^2} \right) Z_0 - \frac{27}{512} \frac{p_{0R}^2}{T_{0R}^3} = 0 \qquad \text{cfv.35}$$

where

$$p_{0R} = \frac{P}{P_c}$$

$$T_{0R} = \frac{T}{T_c}$$

$$P_c = critical\ pressure\ for\ air$$

$$T_c = critical\ temperature\ for\ air$$

We need the first derivative to use Newton's method to solve the equation

$$f'(Z_0) = 3Z_0^2 - 2\left(\frac{p_{0R}}{8T_{0R}} + 1 \right) Z_0 + \left(\frac{27}{64} \frac{p_{0R}}{T_{0R}^2} \right) \qquad \text{cfv.36}$$

We can then obtain an iterative solution using

$$Z_{0,i} = Z_{0,i} - \frac{f(Z_0)}{f'(Z_0)} = Z_{0,i} - \frac{Z_0^3 - \left(\frac{p_{0R}}{8T_{0R}} + 1\right)Z_0^2 + \left(\frac{27}{64}\frac{p_{0R}}{T_{0R}^2}\right)Z_0 - \frac{27}{512}\frac{p_{0R}^2}{T_{0R}^3}}{3Z_0^2 - 2\left(\frac{p_{0R}}{8T_{0R}} + 1\right)Z_0 + \left(\frac{27}{64}\frac{p_{0R}}{T_{0R}^2}\right)} \quad \text{cfv.37}$$

Which can be cast in the form

$$Z_{0,i} = Z_{0,i} - \frac{Z_0^3 - mZ_0^2 + nZ_0 - o}{3Z_0^2 - 2mZ_0 + n} \quad \text{cfv.38}$$

where

$$m = \left(\frac{p_{0R}}{8T_{0R}} + 1\right) \qquad n = \left(\frac{27}{64}\frac{p_{0R}}{T_{0R}^2}\right) \qquad o = \frac{27}{512}\frac{p_{0R}^2}{T_{0R}^3} \quad \text{cfv.39}$$

References

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